stichting
mathematisch
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MC

AFDELING ZUIVERE WISKUNDE ZW 9/71 SEPTEMBER

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

COMPACT SETS IN NON-METRIZABLE PRODUCT SPACES

BY

J. VAN DER SLOT

INTRODUCTION

Denote by s the countable infinite topological product of real lines. A result in infinite dimensional topology [1], [2] and [3] states that whenever C is a compact subset of s, then s\C is homeomorphic with s. In this note our aim is to show that for products of more than countable many real lines the situation is completely different; in fact we have the following result: Let C_1 and C_2 be compact subsets of an <u>uncountable</u> product P of real lines. Then the condition C_1 homeomorphic with C_2 is necessary and sufficient in order that the complements $P\backslash C_1$ and $P\backslash C_2$ are homeomorphic. We also study some generalizations of this result which might be of interest.

The author is indebted to A. Verbeek and M. Hušek for some valuable remarks.

P stands for some uncountable product of real lines. If $P = \Pi\{R_\alpha \mid \alpha \in A\}$ where $R_\alpha = R$ for $\alpha \in A$ then π_α denotes the natural projection of P onto R_α; if $p = (p_\alpha)_\alpha \in P$ then $\Sigma(p) \subset P$ is defined by $\Sigma(p) = \{x = (x_\alpha)_\alpha \in P | x_\alpha \neq p_\alpha$ for at most countably many $\alpha\}$.

The following lemma is due to Corson [4]. See also [5] page 98.

LEMMA

If f is a continuous map of $\Sigma(p)$ into $\mathbb{R}$ then f can be extended uniquely to a continuous function over P.

This lemma gives us the necessary machinery to prove the only if part of our statement. It gives us the following theorem:

THEOREM 1

If C_1 and C_2 are compact subsets of P and i is a homeomorphism of $P \setminus C_1$ onto $P \setminus C_2$ then there is an autohomeomorphism i^* of P onto P which is an extension of i and which carries C_1 onto C_2.

PROOF

First we notice that if E is an arbitrary compact set in P then for some $p = (p_\alpha)_\alpha \in P$ we have $E \subset P \setminus \Sigma(p)$. Indeed, for each α we have $\pi_\alpha E \neq \mathbb{R}_\alpha$, thus there is $p_\alpha \in \mathbb{R}_\alpha \setminus \pi_\alpha E$ for each α. If $p \in P$ is defined such that the α'th coordinate is p_α, then $E \subset P \setminus \Sigma(p)$.

Now, if i is a homeomorphism of $P \setminus C_1$ onto $P \setminus C_2$ then by lemma 1 the composition maps $i_\alpha = \pi_\alpha \circ i : P \setminus C_1 \to \mathbb{R}_\alpha$ can be extended to mappings $i_\alpha^*: P \to \mathbb{R}_\alpha$; thus $i^*: P \to P$ defined by $(i^*(x))_\alpha = i_\alpha^*(x)$ $(\alpha \in A)$ is a continuous extension of i taking C_1 into C_2. In the same way $j = i^{-1}: P \setminus C_2 \to P \setminus C_1$ can be extended to $j^*: P \to P$. Thus the mappings $j^* \circ i^*$ and $i^* \circ j^*$ are the identities on P, since both $P \setminus C_2$ and $P \setminus C_1$ are dense. Hence i^* satisfies our hypotheses.

REMARK

The lemma states that P is homeomorphic with the Hewitt-real-compactification of $\Sigma(p)$: $P = \upsilon(\Sigma(p))$ (see [5]). If $E \subset P$ is compact then $P = \upsilon(P \setminus E)$ i.e., E is the remainder of $P \setminus E$ in the Hewitt-real-compactification of $P \setminus E$.

The "if" part of the main result can now be stated as follows. For the proof we use the methods of Klee [7] and Anderson [1]. The present setting was suggested to me by A. Verbeek (oral communucation).

THEOREM 2

If C_1 and C_2 are compact subsets of P and h is a homeomorphism of C_1 onto C_2 then h can be extended to an autohomeomorphism of P onto P. In particular it follows that $P \setminus C_1$ is homeomorphic with $P \setminus C_2$.

PROOF

Let $P = \Pi\{P_\alpha \mid \alpha \in A\}$ where $P_\alpha = \Pi\{\mathbb{R}_n^\alpha \mid n = 1, 2, \ldots\}$ and $\mathbb{R}_n^\alpha = \mathbb{R}$ $(\alpha \in A, n=1,2,\ldots)$, and let $C = C_1 \cup C_2$. For each α there is an autohomeomorphism ϕ_α: $P \to P$ such that for some $n(\alpha) \in N$ $\pi_{n(\alpha)}(\phi_\alpha C)$ is a single element ([2] page 779, lemma 6.1.).

Now, let ϕ: $P \to P$ be defined by $(\phi(x))_\alpha = \phi_\alpha(x)$ $(\alpha \in A)$, then ϕ is a homeomorphism between P and the product of two copies P' and P" of P such that $\pi_{P'}\phi(C)$ consists of a single element. Indeed, let $P' = \Pi\{\mathbb{R}^\alpha_{n(\alpha)} \mid \alpha \in A\}$ and P" be the product of the factors $\mathbb{R}^\alpha_n$ for $n \neq n(\alpha)$ $(\alpha \in A)$. The proof of the theorem thus reduces to the following lemma:

LEMMA

Let p be a point of P; C_1 and C_2 compact subsets and h be a homeomorphism of C_1 onto C_2. Then h_1: $\{p\} \times C_1 \to \{p\} \times C_2$ defined by $h_1(p,x) = (p,h(x))$ can be extended to an autohomeomorphism of $P \times P$ onto $P \times P$.

PROOF

One can suppose that $p = 0 \in P$. Let h^*: $P \to P$ be a continuous extension of h and h^{-1*}: $P \to P$ be a continuous extension of h^{-1}: $C_2 \to C_1$. Such extensions exist because the compact sets C_1 and C_2 are C-embedded in P. For each $x = (x',x'') \in P \times P$ define

$$\psi_1(x',x'') = (x'+x'',x'')$$

$$\psi_2(x',x'') = (x',x''+h^*(x')-x')$$

$$\psi_3(x',x'') = (x'-h^{-1*}(x''),x'').$$

Here + and - stand for usual vector addition in the topological product P. Obviously ψ_1,ψ_2 and ψ_3 are autohomeomorphisms of $P \times P$ onto $P \times P$ and the composition $\psi = \psi_3 \circ \psi_2 \circ \psi_1$ satisfies the hypothesis of the lemma. Indeed, if $x'' \in C_1$ then $\psi(0,x'') = (\psi_3 \circ \psi_2)(x'',x'') = (x''-h^{-1}h(x''),h(x'')) = (0,h(x''))$. This completes the proof.

From Th. 1 and 2 we deduce:

COROLLARY

Two compact subsets of P are homeomorphic if and only if their complements in P are homeomorphic.

EXAMPLE

Let A be of cardinality of the continuum and $P = \Pi\{\mathbb{R}_\alpha \mid \alpha \in A\}$, $\mathbb{R}_\alpha = \mathbb{R}$ $(\alpha \in A)$. If $p \in P$ and C is an arc lying in P then the previous result implies $P \setminus \{p\} \not\cong P \setminus C$. We also have $P \setminus \{p\} \not\cong P$.

GENERALIZATIONS

Instead of considering products of real lines we can also consider uncountable products of intervals [0,1]. To obtain the corresponding theorem in this case, one has to introduce the concept of partial deficiency [3] and generalize it for uncountable products.

We say that a closed subset C of an uncountable product of $\underline{m}$ unit intervals I ($\underline{m}$ infinite cardinal) has a complete partial deficiency if $\overline{\pi_\alpha C}$ is contained in (0,1) for $\underline{m}$ indices α.

Now we have the following generalization:

THEOREM 3

If two closed subsets C_1 and C_2 of an uncountable product K of closed intervals are of complete partial deficiency, then C_1 and C_2 are homeomorphic if and only if $K \setminus C_1$ and $K \setminus C_2$ are homeomorphic.

PROOF

First, note that if C_1 and C_2 are completely partial deficient then there is an autohomeomorphism i of K onto itself such that $i(C_1 \cup C_2)$ is completely partial deficient. This can easily be deduced from the corresponding statement in the countable case: the union of two Z-sets in the Hilbert-cube is a Z-set (cf. [2]). Now the proof is reduced to the previous case except that in theorem 3 to prove necessity we have to use e.g. a piecewise linear homeomorphism rather than simple vector addition.

Theorem 3 also remains valid if we consider an uncountable product of open intervals and demand C_1 and C_2 to be closed and C-embedded. The condition C-embedded is essential to obtain the extensions h^* and h^{-1*} in Klee's lemma.

The second generalization consists in considering σ-compact subsets rather than compact sets. Theorem 1 remains valid if we replace compact by σ-compact.

Indeed, if $C = \cup\{C_i \mid i = 1, 2, \ldots\}$ where $C_i \subset P = \Pi\{\mathbb{R}_\alpha \mid \alpha \in A\}$ then write the index set A as a countable union of disjoint uncountable sets A_i, $i = 1, 2, \ldots$ and for each $\alpha \in A_i$ $i = 1, 2, \ldots$ define $p_\alpha \in \mathbb{R}_\alpha$ such that $p_\alpha \notin \pi_\alpha C_i$. The point $p \in P$ whose α'th coordinate is p_α $(\alpha \in A)$ satisfies the condition $C \subset P \setminus \Sigma(p)$.

The above argument implies that for two σ-compact C_1 and $C_2 \subset P$ the condition $P \setminus C_1$ homeomorphic with $P \setminus C_2$ necessarily implies that both or neither one of C_1 and C_2 must be closed and C-embedded. However, there exists a closed countable discrete subset of P which is not C-embedded in P (see [6]). Thus there exist two closed countable discrete subspaces of P such that their complements in P are not homeomorphic.

However, we do not know if for two closed and C-embedded σ-compact subsets C_1 and C_2 the condition $C_1 \cong C_2$ necessarily implies $P \setminus C_1 \cong P \setminus C_2$.

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